

$1/f^\alpha$ Noise in Spectral Fluctuations of Quantum Systems

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The power law $1/f^\alpha$ in the power spectrum characterizes the fluctuating observables of many complex natural systems. Considering the energy levels of a quantum system as a discrete time series where the energy plays the role of time, the level fluctuations can be characterized by the power spectrum. Using a family of quantum billiards, we analyze the order-to-chaos transition in terms of this power spectrum. A power law $1/f^\alpha$ is found at all the transition stages, and it is shown that the exponent α is related to the chaotic component of the classical phase space of the quantum system.

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Of the different features which characterize complex physical systems, perhaps the most ubiquitous, interesting, and puzzling is the presence of $1/f^\alpha$ noise [1] in fluctuating physical variables, i.e., the Fourier power spectrum $S(f)$ behaves as $1/f^\alpha$ in terms of the frequency f . This kind of noise has been detected in condensed matter systems, traffic engineering, DNA sequence, quasar emissions, river discharge, human behavior, heartbeat and dynamic images, among many others. Despite this ubiquity, there is no universal explanation about this phenomenon. It does not arise as a consequence of particular physical interactions, but it is a generic manifestation of complex systems.

Recently, it was conjectured that *the energy spectra of chaotic quantum systems are characterized by $1/f$ noise* [2]. The original idea was that the sequence of discrete energy levels in a quantum system can be considered as a discrete time series, where the energy plays the role of time. In that case, the energy level fluctuations can be studied using traditional methods of time series analysis, like the study of the power spectrum. When the idea was applied to typical chaotic quantum systems, the power spectrum showed a very accurate $1/f$ behavior [2]. Hence, chaotic quantum systems can be added to the long list of complex natural systems which exhibit $1/f$ noise. However, this new point of view also raises new questions. Is this a consequence of the universal behavior of fluctuations in chaotic quantum systems? What happens in quantum systems which are neither fully chaotic nor fully regular? In this Letter we try to find some answers using a quantum billiard to study the power spectrum in the order-to-chaos transition. As shown below, the ubiquitous $1/f^\alpha$ noise appears at all the transition stages, with the exponent smoothly decreasing from $\alpha = 2$ in a regular system to $\alpha = 1$ in a chaotic system. This is quite a remarkable result indeed, since it contradicts the predictions of the strict semiclassical limit [3].

The concept of quantum chaos, or wave chaos in more general terms [4], has no unique precise definition as yet, but definitely can be described as quantum or wave like signatures of classical chaos. It is well known that there is a relationship between the energy level fluctuation properties of a quantum system and the dynamics of its classical limit. Classically integrable systems give rise to uncorrelated adjacent energy levels, which are well described by Poisson statistics [5]. In contrast, spectral fluctuations of a quantum system whose classical limit is fully chaotic (ergodic) show a strong repulsion between energy levels and follow the predictions of random matrix theory (RMT) [6,7]. In practice, quantum systems without classical limit are assumed to be chaotic when their fluctuations coincide with RMT predictions.

The essential feature of chaotic energy spectra is the existence of level repulsion and correlations (leading to strong spectral rigidity), i.e., the spacing of two adjacent levels is unlikely to deviate much from the mean spacing. This property is similar to the *antipersistence* characteristic of some time series [1]. Antipersistence, with different intensity degrees, appears in time series with $1/f^\alpha$ noise, with $1 < \alpha < 2$. Could the analogue level repulsion feature be also associated to $1/f^\alpha$ noise?

To study the spectral fluctuations of quantum systems we follow the method introduced in [2]. We use the statistic δ_n defined by

$$\delta_n = \sum_{i=1}^n (s_i - \langle s \rangle) = \epsilon_{n+1} - \epsilon_1 - n, \quad (1)$$

where ϵ_i are the unfolded energy levels, $s_i = \epsilon_{i+1} - \epsilon_i$, and $\langle s \rangle = 1$ is the average value of s_i . Thus δ_n represents the fluctuation of the n th excited state. Formally δ_n is similar to a time series where the level order index n plays the role of a discrete time. Therefore the statistical behavior of level fluctuations can be investigated studying the power spectrum $S(k)$ of the signal, given by

$$S(k) = \left| \frac{1}{\sqrt{M}} \sum_{n=1}^M \delta_n \exp\left(\frac{-2\pi i k n}{M}\right) \right|^2, \quad (2)$$

where M is the size of the series and $f = 2\pi k/M$ plays the role of a frequency.

To investigate the behavior of $S(k)$ in the mixed regime between integrability and chaos, we analyze it in the Robnik billiard [8]. Quantum billiards are considered as a paradigm in quantum chaos. They have a discrete spectrum with an infinite number of eigenvalues, and therefore it is possible to reach high statistical precision by computing a large number of them. Furthermore, they can also be studied experimentally [4,9].

The boundary of the Robnik billiard is defined as the set of points w in the complex plane \mathbb{C} which satisfy the equation $w = z + \lambda z^2$, where $|z| = 1$ and λ is the deformation parameter. It has been shown [8] that this billiard exhibits a smooth transition from the integrable case ($\lambda = 0$) to an almost chaotic case ($1/4 \leq \lambda \leq 1/2$). In order to obtain a smooth analytic boundary, λ must lie in the interval $[0, 1/2)$. The Robnik billiard is one of the best systems to investigate the order-to-chaos transition [8,10,11]. Compared to other quantum billiards, it has the advantage that there are no bouncing ball orbits. For small values of λ the billiard is a typical Kol'mogorov-Arnol'd-Moser system, whereas for larger values of λ only one chaotic region dominates the phase space with only few stability islands covered with invariant tori. The total area in the bounce map (Poincaré surface of section) of these invariant tori decreases monotonically with λ and becomes negligible when the shape of the billiard becomes non-convex, for $\lambda > 1/4$. For $\lambda = 1/2$ it has been shown rigorously by Markarian [12] that the billiard is ergodic.

The quantum energy levels E_n of the Robnik billiard are numerically calculated by solving the stationary Schrödinger equation of a free particle whose wave func-

tion $\psi(w)$ is zero at the boundary of the billiard. The billiard has reflection symmetry with respect to the real axis, so there are two types of states: those with even parity $\psi(w) = \psi(w^*)$ and those with odd parity $\psi(w) = -\psi(w^*)$; odd and even parity states must be treated separately [10,11]. For each symmetry, our calculation uses approximately 80 000 basis states, giving good eigenvalues for about 65 000 levels for $\lambda = 0$ and about 30 000 levels for $\lambda = 0.5$.

Figure 1 shows the energy level fluctuations of the Robnik billiard given by δ_n . It illustrates the effect of level repulsion in the order-to-chaos transition, and its relationship with the antipersistence of δ_n considered as a time series. For the regular system ($\lambda = 0$), the levels are uncorrelated and therefore δ_n is neither persistent nor antipersistent. As λ increases the system becomes more chaotic, and δ_n looks like a typical antipersistent series in the almost chaotic region for $\lambda > 1/4$.

Figure 2 shows the power spectrum of δ_n obtained from the energy levels E_n of the Robnik billiard for several values of λ . The contour of the billiard is shown as an inset in each panel. We calculate an ensemble average of $S(k)$ in order to reduce statistical fluctuations and clarify its main trend. The average $\langle S(k) \rangle$ is calculated with 25 sets of 256 consecutive high energy levels of odd parity. It is clearly seen that for each value of λ it follows the scaling law

$$\langle S(k) \rangle \sim \frac{1}{k^\alpha}, \quad (3)$$

where α depends on λ . In fact the fit of $\langle S(k) \rangle$ to the power law $1/k^\alpha$ is excellent. In all cases the error in the linear regression is less than 3%. For $\lambda = 0$ (integrable case) the exponent is $\alpha = 1.98$, as expected for uncorrelated energy levels. As λ increases the exponent α decreases and be-

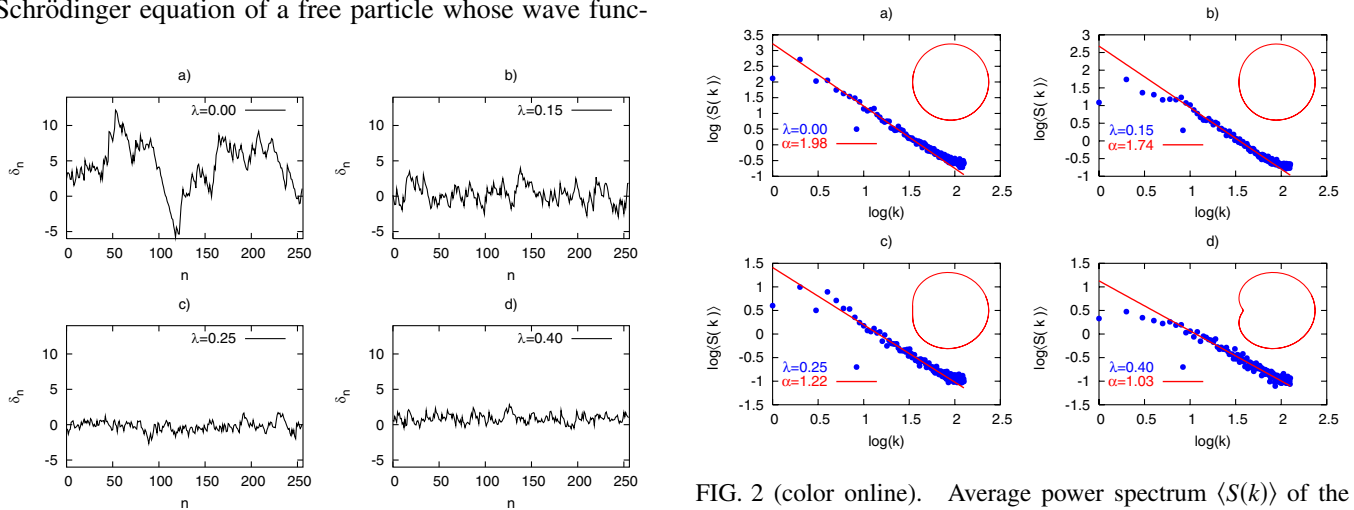


FIG. 1. Plot of the statistic δ_n for a set of 256 consecutive energy levels of odd parity in the Robnik billiard, for several values of the deformation parameter λ .

FIG. 2 (color online). Average power spectrum $\langle S(k) \rangle$ of the statistic δ_n for the odd parity energy levels corresponding to the shapes of the Robnik billiard inserted in the figures. The four values of the deformation parameter λ are the same as in Fig. 1. The solid (red) line is the best fit to the power law $1/k^\alpha$.

comes $\alpha \approx 1$ for $\lambda \approx 1/2$. Thus, α may serve as a measure of the chaoticity of the system, since it changes from $\alpha = 2$ for regular systems to $\alpha = 1$ for chaotic ones.

It is worthwhile to compare this result with the behavior of more conventional statistics, like the nearest neighbor spacing distribution $P(s)$, which measures short range correlations, and the Dyson $\Delta_3(L)$ statistic appropriate for correlations of length L [13]. Figure 3 displays $P(s)$ for several values of λ . At $\lambda = 0$ the histogram follows the predicted curve for regular systems (Poisson limit). For $\lambda = 0.15$, $P(s)$ deviates from Poisson toward the RMT limit. As we shall see below, this behavior of $P(s)$ reflects that the underlying classical dynamics is neither regular nor ergodic (chaotic). Finally, for $\lambda = 0.25$ and $\lambda = 0.4$ the system exhibits short range correlations characteristic of chaotic systems (RMT limit). Figure 4 shows the spectral average $\langle \Delta_3(L) \rangle$ for energy intervals of length L ranging from $L = 2$ to $L = 50$, for several values of λ . The spectral average is calculated using 25 sets of L consecutive high energy levels to avoid, as far as possible, the influence of short periodic orbits. The evolution of this statistic with λ is analogous to that of $P(s)$. When $\lambda = 0$, $\langle \Delta_3(L) \rangle$ falls near the Poisson prediction for regular systems, and for $\lambda = 0.25$ and 0.4 it is almost indistinguishable from the RMT prediction for chaotic systems. Therefore, $P(s)$ and $\langle \Delta_3(L) \rangle$ have a smooth behavior in terms of λ . Nevertheless, they move faster than δ_n toward the RMT limit as λ increases. For instance, both $P(s)$ and $\langle \Delta_3(L) \rangle$ coincide with RMT predictions for $\lambda = 0.25$, while δ_n still points to an intermediate regime between regularity and chaos.

Let us now compare the evolution of the parameter α as a function of λ with the fraction ρ_1^{cl} of regular classical trajectories in the phase space, and with the Brody parameter ω [14]. This is an *ad hoc* parameter, without any known physical meaning, which quantifies to some extent the

chaoticity of the system. It nevertheless captures well the important feature of fractional power law level repulsion, that is the behavior of $P(s)$ at small s [10,11]. For $\omega = 0$ we get the Poisson distribution of regular systems, and at the other extreme, $\omega = 1$, we obtain the Wigner distribution predicted by RMT for chaotic systems. Figure 5 shows the behavior of α , ω , and ρ_1^{cl} . There is a clear correlation among these three variables, although the transition is smoother for ω and especially for α than for ρ_1^{cl} . However, while the fraction of regular classical trajectories is almost zero near $\lambda = 0.25$, the power spectrum, and to a lesser extent the $P(s)$ distribution, indicate an intermediate situation between regular and chaotic motion. This clearly shows that ω and α are not only functions of ρ_1^{cl} , but depend on finer details of the underlying classical mechanics as well.

It is well known that in the strict semiclassical limit the quantum eigenstates of a quantum system with generic (mixed) classical dynamics can be classified as regular and irregular following the original proposition of Percival [15]. This has been further developed and raised to a principle of uniform semiclassical condensation of Wigner functions of the eigenstates [16]. From this it follows that in the strict semiclassical limit the regular and irregular level sequences are statistically independent, but for themselves, have Poisson or RMT level statistics, respectively. This theory has been excellently confirmed in hard numerical calculations [17]. Nevertheless, if the system is not deep enough in the semiclassical regime one can see substantial deviations from such a behavior, manifested in the fractional power law level repulsion [10,11] in $P(s)$ at small s . A similar recent analysis [3] has demonstrated that in the strict semiclassical limit we should not expect a power law behavior for the power spectrum but something more complicated. Therefore it is quite an unexpected result of the present Letter that the power spectrum $S(k)$ is a power law at all k in mixed systems at low energies. Indeed, when going sufficiently deep into the semiclassical

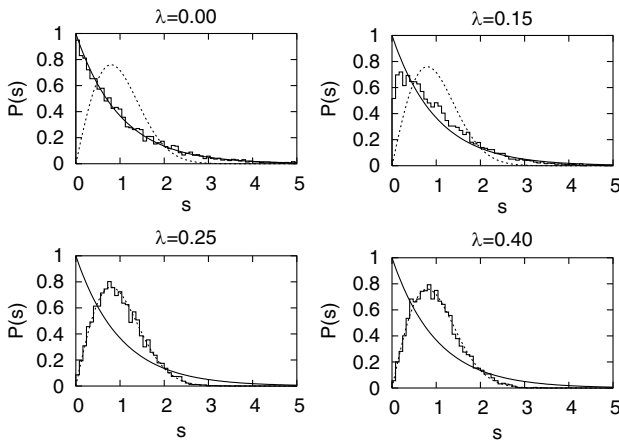


FIG. 3. Nearest neighbor spacing distributions for the spectra of four different shapes of the Robnik billiard. The solid line is the Poisson distribution and the dotted line corresponds to the Wigner distribution predicted by RMT.

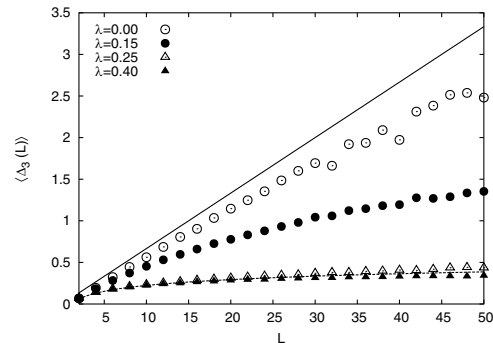


FIG. 4. Spectral average $\langle \Delta_3(L) \rangle$ values calculated numerically for the Robnik billiard using the same deformation parameters as in previous figures. The results of the Poisson statistics (solid line) and RMT (dotted line, superimposed to the triangles) are also displayed.

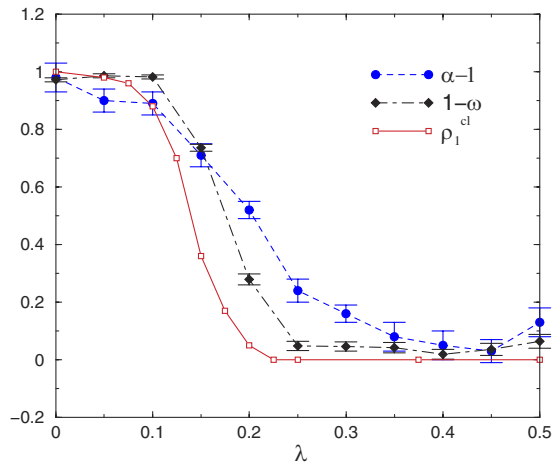


FIG. 5 (color online). Behavior of the power spectrum exponent α , the Brody parameter ω , and the fraction ρ_1^{cl} of regular classical trajectories of the Robnik billiard as functions of the deformation parameter λ .

regime, the theory of Ref. [3] should be expected and confirmed.

In conclusion, the analogy between quantum energy spectra and time series opens a new and fruitful perspective on the universal properties of quantum level fluctuations. The δ_n function gives the level fluctuations, and its power spectrum $S(k)$ is an intrinsic characteristic of the quantum system. The important point is that it exhibits a power law behavior, similar to the well known $1/f^\alpha$ noise found in many complex systems.

In the order-to-chaos transition, the chaoticity of a quantum system is usually qualitatively assessed by how close to Poisson or RMT its fluctuation properties are. In the present power spectrum approach, the exponent changes smoothly from $\alpha = 2$ for a regular system to $\alpha = 1$ for a chaotic system. Contrary to the Dyson $\Delta_3(L)$ statistic, that must be plotted for different values of L , the exponent α quantifies the chaoticity of the system in a single parameter. Moreover, α has a physical meaning. It is a natural measure of the fluctuation properties of a quantum system through the power spectrum, and provides an intrinsic quantitative measure of the regular and chaotic dynamical features.

The origin of the universal power law behavior $S(k) \sim 1/k^\alpha$ is now understood in the integrable case ($\alpha = 2$) and in the fully chaotic case ($\alpha = 1$) on the basis of RMT [18] and semiclassical periodic orbit theory [3]. The origin of

the $1/f^\alpha$ power law in the mixed regime still remains as an important open problem.

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